Question 1. Consider $\mathbb{R}$ with the co-finite topology, say τ. Which of the following sequences are convergent and what are their limits?

(i) $\{a_n\}_n = \{1, 2, 3 \dots\}$, i.e., $a_n = n$

(ii) $\{a_n\}_n = \{1, 1, 2, 1, 3, 1 \dots\}$, i.e., $a_{2n-1} = n$ and $a_{2n} = 1$)

(iii) $\{a_n\}_n = \{1, 2, 1, 2, 1, 2 \dots\}$, i.e., $a_{2n-1} = 1$ and $a_{2n} = 2$)

Answer:

(i) The sequence $\{a_n\}_n$ is convergent and converges to every points in $\mathbb{R}$. Let $x \in \mathbb{R}$ and U_x be an open set in τ containing x. Then $U_x = \mathbb{R} \setminus \{x_1, x_2, \dots, x_n \,|\, x_i \neq x \text{ and } x_i \in \mathbb{R}\}$. Let $m = [\max\{x_1, x_2, \dots, x_n\}] + 1$, then $a_i \in U_x$ for all $i \geq m$. Therefore, the sequence $\{a_n\}_n$ converges to every points in $\mathbb{R}$.

(ii) The sequence $\{a_n\}_n$ is convergent and converges to 1. Let U be an open set in τ containing 1. Then $U = \mathbb{R} \setminus \{x_1, x_2, \dots, x_n \,|\, x_i \neq 1 \text{ and } x_i \in \mathbb{R}\}$. Let $m = [\max\{x_1, x_2, \dots, x_n\}] + 1$, then $a_i \in U_x$ for all $i \geq 2m - 1$. Therefore, the sequence $\{a_n\}_n$ converges to 1.

Let $1 \neq x \in \mathbb{R}$ and V be an open set $\mathbb{R} \setminus \{1\}$ in τ containing x. Since $1 \notin V$, $a_{2i} \notin V$. Thus, the sequence $\{a_n\}_n$ will not converge to x.

(iii) The sequence $\{a_n\}_n$ is not convergent. Let $x \in \mathbb{R}$. Let

$$U = \begin{cases} \mathbb{R} \setminus \{1\} & \text{if } x \neq 1 \\ \mathbb{R} \setminus \{2\} & \text{if } x = 1. \end{cases}$$

Then U is an open set in τ containing x. Now, either $1 \notin U$ or $2 \notin U$, implies that, either $a_{2i-1} \notin U$ or $a_{2i} \notin U$. Thus, the sequence $\{a_n\}_n$ will not converge to x.

Question 2. Prove that any second countable space is first countable. Is the converse true? Justify.

Answer: A space X is called *first countable* if each $x \in X$ has a countable neighborhood basis. A space X is called *second countable* if there exists a countable basis for the topology of X. Thus, by definition, every second countable space is first countable: if $\{U_i\}_{i=1}^{\infty}$ is a countable basis for the topology of a space X, then for each $x \in X$, the sets $\{U_i \,|\, x \in U_i\}$ form a countable neighborhood basis of X.

The converse is not true. Let X be any uncountable set. The function d given by $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$ is a metric, and the corresponding topology on X is discrete. Thus every point of X is an open set, which implies that (X, d) is not second countable (because X itself is assumed to be uncountable). But it is first countable: for a given $x \in X$, the open sets $\{\{x\}, X\}$ form a countable neighborhood basis of x.

Question 3. Show that $\mathbb{R}$ with the co-finite topology is not regular. Prove also that $\mathbb{R}$ with the co-countable topology is not regular.

Answer: For $x \in \mathbb{R}$, the complement of $\mathbb{R} \setminus \{x\}$ is finite (resp., countable). Therefore, $\mathbb{R} \setminus \{x\}$ is a open set in co-finite and co-countable topology and hence $\{x\}$ is a closed set in both topology. Therefore, every one-point sets are closed in both topology.

But, there are no two disjoint open sets in $\mathbb{R}$ with the co-finite and co-countable topology. Let U and V be two disjoint open sets in $\mathbb{R}$ with the co-finite (resp., co-countable) topology. Then $V \subset \mathbb{R} \setminus U$ which is finite (resp., countable). Therefore V is finite (resp., countable). Since $\mathbb{R}$ is uncountable, $\mathbb{R} \setminus V$ will not be finite (resp., countable) and hence V will not be a open set in $\mathbb{R}$ with the co-finite (resp., co-countable) topology. Hence, $\mathbb{R}$ is not regular with the co-finite and co-countable topology

Question 4. If x and y are distinct points of a regular space (X, τ), show that there exist open sets U and V such that $x \in U, y \in V$ and $\bar{U} \cap \bar{V} = \emptyset$.

Answer: Since one-point sets are closed in a regular space, $\{y\}$ is a closed set. Now, by the definition, there are disjoint open sets containing x and $\{y\}$, i.e., there exist open sets O_x containing x and O_y containing $\{y\}$ such that $O_x \cap O_y = \emptyset$. Now, by Lemma 31.1 of Munkress Topology book, there exist open sets U and V such that $x \in U \subset \bar{U} \subset O_x$ and $y \in V \subset \bar{V} \subset O_y$. Since $O_x \cap O_y = \emptyset$, $\bar{U} \cap \bar{V} = \emptyset$.

Question 5. Prove that any open connected set in $C[0,1]$ is polygonally connected.

Here $C[0,1]$ is the space of real valued continuous function on $[0,1]$ with the metric: $d(f,g) = \sup\{|f(x) - g(x)| : 0 \leq x \leq 1\}$. A polygonal path is a path made up of a finite number of line segments.

Answer: *Claim.* Every open ball in $C[0,1]$ is convex.

Let $B_r(f) = \{g : d(f,g) \leq r\}$ be an open ball in $C[0,1]$. Let $g, h \in B_r(f)$. Then, for $0 \leq t \leq 1$, $d(f, tg+(1-t)h) = \sup\{|f(x) - (tg + (1-t)h)(x)| : 0 \leq x \leq 1\} = \sup\{|t(f(x) - g(x)) + (1-t)(f(x) - h(x))(x)| : 0 \leq x \leq 1\} \leq t\sup\{|f(x) - g(x)| : 0 \leq x \leq 1\} + (1-t)\sup\{|f(x) - h(x)| : 0 \leq x \leq 1\} \leq tr + (1-t) = r$. Therefore, $B_r(f)$ is convex.

Let U be a open connected set in $C[0,1]$. We have to prove that U is polygonally connected. Let $f \in U$ and $V \subset U$ is a collection of all elements in U, which are joined with f by a polygonal path.

Claim. V is an open subset of U.

Let $g \in V \subset U$. Since U is an open set, there is a ball $B_r(g) \subset U$. Since $B_r(g)$ is convex and f and g are joined by a polygonal path, every elements of $B_r(g)$ are joined with f by a polygonal path and hence $B_r(g) \subset V$. Therefore, V is an open set in U

Claim. $U \setminus V = \emptyset$.

If $U \setminus V \neq \emptyset$ then let $h \in U \setminus V \subset U$. Since U is an open set, there is a ball $B_t(h) \subset U$. If $B_t(h) \cap V \neq \emptyset$ then there is a polygonal path between f and h as $B_t(h)$ is convex, which is not possible. Therefore, $B_t(h) \cap V = \emptyset$, i.e., $B_t(h) \subset U \setminus V$. Thus, $U \setminus V$ is an open in U. This contradicts the fact that U is connected. Therefore, $U \setminus V = \emptyset$, i.e., $V = U$.

This proves that U is polygonnaly connected.

Question 6. Let A be a subgroup of $\mathbb{R}$ under addition. Show that either A is dense in $\mathbb{R}$ or else the subspace topology of A is the discrete topology.

Answer: Since A is a subgroup, $0 \in A$. If $A = \{0\}$ then subspace topology of A is the discrete topology. Now, we assume that $A \neq \{0\}$.

Let $r = \inf\{a > 0 : a \in A\} \neq 0$. If $r \notin A$ then there is a $x \in A$ such that $r < x < r + r/2$ and there is a $y \in A$ such that $r < y < x$ as $r = \inf\{a > 0 : a \in A\}$ and $r \notin A$. Therefore, $x - y < r/2$ and $x - y \in A$ as A is a additive subgroup of $\mathbb{R}$. This contradicts that $r = \inf\{a > 0 : a \in A\}$. Therefore, $r \in A$ and hence $\{0, \pm r, \pm 2r, \dots\} \subset A$.

claim. $A = \{0, \pm r, \pm 2r, \dots\}$.

Let $x \in A \setminus \{0, \pm r, \pm 2r, \dots\}$. Then $kr < x < (k+1)r$ for some $k \in \mathbb{Z}$. Therefore, $0 < x - kr < r$ and $x - kr \in A$. This contradicts the fact that $r = \inf\{a > 0 : a \in A\}$. Therefore, $A = \{0, \pm r, \pm 2r, \dots\}$. Thus, subspace topology of A is the discrete topology.

claim. If $\inf\{a > 0 : a \in A\} = 0$ then A is dense in $\mathbb{R}$.

Let $0 \neq x \in A$. Let $y \in \mathbb{R} \setminus A$ and $(y - t, y + t)$ be an arbitrary open interval containing y. Since $\inf\{a > 0 : a \in A\} = 0$ and $A \neq \{0\}$, there exists $z \in A$ such that $0 < z < t$. Therefore, $lz \in A$ for all $l \in \mathbb{Z}$. Now, there exists a $k \in \mathbb{Z}$ such that $kz < y < (k+1)z$. Therefore, $y - t < kz < y$ and hence A is dense in $\mathbb{R}$.